Analysis

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The Real Numbers

The Irrationality of $\sqrt{2}$

Theorem 0.0.1 *The is no rational number whose square is 2*

*Proof:*assume $\sqrt{2} = p/q$ for some $p, q \in \mathbb{Z}$ with $gcd(p, q) = 1$

$\Rightarrow p^2 = 2q^2$
$\Rightarrow p^2$ is even, so p itself is even, say $p = 2k$
$\Rightarrow 4k^2 = 2q^2$ so $2k^2 = q^2$
$\Rightarrow q^2$ is even, which bring a contradiction becuase we assumed that $gcd(p, q) = 1$

Some Preliminaries

Definition 0.0.1 (set) *A set is any collection of objects. These objects are referred to as* <u>*elements*</u> *of the set*

Set-Theoretic Notation:

- $A \cup B$: A union B
- $A \cap B$: A intersect B
- A^c: $\{x \in \Omega : x \notin A\} \Rightarrow$ complement of A
- $w \in \Omega$: w is an element of Ω;
- $A \subseteq \Omega$: A is an subset of Ω
- $B \supseteq A$: B contains A (its the same of the previous);
- The set $\emptyset$ is called <u>empty set</u>
- $\bigcup_{n=\mathbb{N}} A_n \Rightarrow A_1 \cup A_2 \cdots$

Theorem 0.0.2 (De Morgan's Laws) *Let A and B be subsets of $\mathbb{R}$, then:*
$(A \cap B)^c = A^c \cup B^c$ *and* $(A \cup B)^c = A^c \cap B^c$

Proof: We begin by showing that $(A \cap B)^c \subseteq A^c \cup B^c$.

Suppose $x \in (A \cap B)^c$, which means that $x \notin (A \cap B)$. Therefore, $x \notin A \cup B$, which means that $x \in A^c \cup B^c$. Hence, $(A \cap B)^c \subseteq A^c \cup B^c$.

Our proof is now halfway done. To complete it we show the opposite subset inclusion. First we begin with an element x in the set $A^c \cup B^c$, which means that x is an element of A^c or that x is an element of B^c. Thus x is not an element of a least one of the sets A or B. So, x cannot be an element of both A and B. This means that x is an element of $(A \cap B)^c$. Therefore, we have proved the law.

Definition 0.0.2 (Function) *Given two sets A, B, a function from A to B is a rule or mapping that takes each element $x \in A$ and associates with it a single element of B. In this case, we write $f : A \to B$. Given an element $x \in A$, the expression $f(x)$ is used to represent the element of B associated with x by f. The set A is called the domain of f. The range of f is not necessarily equal to B but refers to the subsets of B given by $\{y \in B : y = f(x) \text{ for some } x \in A\}$. That is, the set of all f-images of all the elements of A is known as the range of f. Thus, range of f is denoted by $f(A)$. B is the co-domain.*

This definition of function is more or less the one proposed by Peter Lejeune Dirichlet (1805-1859) in the 1830s.

Absolute Value

Definition 0.0.3 (Absolute Value)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Lemma 0.0.1 $|x| = max\{x, -x\}$

Proof:

First case:

$$\begin{aligned} x > 0 &\Rightarrow -x \leq 0 \\ &\Rightarrow -x \leq x \\ &\Rightarrow max\{-x, x\} = x = |x| \end{aligned}$$

Second case:

$$\begin{aligned} x < 0 &\Rightarrow -x > 0 \\ &\Rightarrow -x > x \\ &\Rightarrow max\{-x, x\} = -x = |x| \end{aligned}$$

Definition 0.0.4 (Product rule)

$$|xy| = |x| \cdot |y|$$

Proof:

- If $x > 0, y > 0$, then by def. $|xy| = xy$ and by def. $xy = |x||y|$;
- if $x = 0, y = 0$ it is obvious that is true: $0 = 0$;
- If $x < 0, y > 0$, then $|xy| = (-x)y$ which by def. $(-x) = |x|, y = |x|$, therefore $(-x)y = |x||y|$;
- If $x > 0, y > 0$ same way of the previus;
- If $x < 0, y < 0$, then $|xy| = (-x)(-y) = |x||y|$

Definition 0.0.5 (quotient rule)

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

where $y \neq 0$

Proof:

- if $x = 0, y > 0$, then $\left|\frac{0}{y}\right| = \frac{0}{y} = 0$;
- same for $x = 0, y < 0$;
- $x < 0, y > 0$, then $\left|\frac{x}{y}\right| = \frac{-x}{y}$ by def. $\Rightarrow \frac{|x|}{|y|}$;
- the same logic for $x < 0, y < 0$ and $x > 0, y < 0$

Inequalities

Lemma 0.0.2

$$|x| \leq a \Leftrightarrow -a \leq x \leq a$$

Proof:

$$\begin{aligned} |x| \leq a &\Rightarrow max\{-x, x\} \leq a \\ &\Rightarrow -x \leq a, x \leq a \\ &\Rightarrow -a \leq x \leq a \end{aligned}$$

Theorem 0.0.3 (Triangle inequality)

$$|x + y| \leq |x| + |y|$$

Proof:

- if $x + y > 0$, then $|x + y| = x + y \leq |x| + y$ by lemma 1.2.1, which is the same for $y \leq |y|$ therefore $|x + y| \leq |x| + |y|$;
- if $x + y < 0$, then $|x + y| = -x - y \leq |x| + |y|$ by lemma 1.2.1

Hence, $|x + y| = max\{x + y, -x - y\} \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$

Theorem 0.0.4 (Reverse triangle inequalities)

$$||x| + |y|| \leq |x - y|$$

Proof:

- $|x| = |x + y - y| \leq |x - y| + |y|$ by theorem 1.2.2;
- $|x| - |y| \leq |x - y|$ which is the same as $|y| - |x| \leq |y - x| = |x - y|$;
- $max\{|x| - |y|, |y| - |x|\} = ||x| + |y|| \leq |x - y|$

Theorem 0.0.5 *Two real numbers a, b are equals if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$*

Induction

Induction is used in conjunction with the natural numbers $\mathbb{N}$. The fundamental principle behind induction is that if S is some subset of $\mathbb{N}$ with the property that

(i). S contains 1 and

(ii). whenever S contains a natural number n, it also contains $n + 1$,

then it must be that $S = \mathbb{N}$.

The Axiom of Completeness

Definition 0.0.6 *A set $A \subseteq \mathbb{R}$ is bounded above if there exist a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.*

Similarly, the set A is bounded below if there exists a lower bound $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$

Definition 0.0.7 (least upper bound) *$s \in \mathbb{R}$ is called the least upper bound of $A \subseteq \mathbb{R}$ if it meets the following two criteria:*

(i). s is an upper bound for A;

(ii). if b is any upper bound for A, then $s \leq b$

The least upper bound is also frequently called the supremum of the set A: $s = supA$

Lemma 0.0.3 *if s is an upper bound for A then*

$$s = supA \Leftrightarrow \forall \epsilon > 0 \ \exists a \in A \ s.t. \ s - \epsilon < a$$

Proof:

(1)Let $\epsilon > 0$, then

$$\begin{aligned} s - \epsilon < s \Rightarrow s - \epsilon &\text{ is not an upper bound for } A \\ &\Rightarrow \exists a \in A \text{ s.t. } s - \epsilon < a \end{aligned}$$

(2)Let b be any upper bound for A

$$\begin{aligned} &\text{if } b < s \Rightarrow \epsilon = s - b \text{ there exist } a \in A \text{ s.t} \\ &\Rightarrow b = s - \epsilon < s \end{aligned}$$

This bring a contradiction. Hence, $s \leq b$, which means that $s = supA$

Definition 0.0.8 (greatest lower bound) *$i \in \mathbb{R}$ is called the greatest lower bound of $A \subseteq \mathbb{R}$ if*

(i). i is a lower bound for A

(ii). if l is any lower bound for A then $l \leq i$

Notation:$i = infA$ *(infimum)*

Lemma 0.0.4 *if i is a lower bound for A then*

$$i = infA \Leftrightarrow \forall \epsilon > 0 \ \exists a \in A \ s.t. \ a < i + \epsilon$$

Proof:

(1)Let $\epsilon > 0$

$$\begin{aligned} i < i + \epsilon \Rightarrow i + \epsilon &\text{ cannot be a lower bound for } A \\ &\Rightarrow \exists a \in A \text{ s.t } a < i + \epsilon \end{aligned}$$

(2)Let l be any lower bound for A

$$\text{if } i < l \Rightarrow \epsilon = l - i$$
$$\Rightarrow \exists a \in A \text{ s.t. } l = \epsilon + i > a$$

This is a contradiction, therefore $l \leq i$, which means that $i = infA$

Axiom of Completeness (AoC) 1 *every nonempty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound*

Consequences of completeness

Theorem 0.0.6 (The Archimedean property) *Theorem:*

(i). $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}$ *s.t.* $n > x$

(ii). $\forall y > 0\ \exists n \in \mathbb{N}$ *s.t* $1/n < y$

Proof:(1) We prove the theorem by contradiction. If (1) is not true, then $\mathbb{N}$ is bounded above.

- AoC$\Rightarrow$ $\alpha = sup\mathbb{N}$ exists.
- $\alpha - 1$ is not an upper bound for $\mathbb{N}$.
- There exist $n \in \mathbb{N}$ such that $\alpha - 1 < n$ by lemma 1.3.1$\Rightarrow$ $\alpha < n + 1$
- $n + 1 \in \mathbb{N} \Rightarrow \alpha$ is not an upper bound for $\mathbb{N}$. Contradiction!

(2)

- AoC$\Rightarrow$ $\alpha = inf\mathbb{N}$
- $\alpha + 1$ is not an lower bound for $\mathbb{N}$
- There exist $n \in \mathbb{N}$ such that $n < \alpha + 1$ by lemma 1.3.2
- $n - 1 < \alpha$, which means that α is not a lower bound for $\mathbb{N}$. Contradiction!

But there is another way to prove part (2), and it's using (i):

Let $y > 0$ be arbitrary and set $x = 1/y$. By (i) there exist $n \in \mathbb{N}$ such that $n > x$. Therefore $1/y < n \Rightarrow 1/n < y$

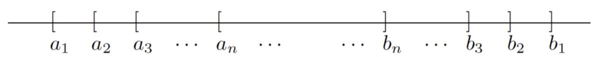
Theorem 0.0.7 (Nested Interval Property) *For each* $n \in \mathbb{N}$, *assume we are given a closed interval* $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. *Assume also that each* I_n *contains* I_{n+1}. *Then, the resulting nested sequence of closed intervals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

has a non-empty intersection; that is,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

*Proof:*Define $A = \{a_n : n \in \mathbb{N}\}$



- Every b_n is an upper bound for A
- AoC$\Rightarrow$ $x = supA \Rightarrow x \leq b_n$ by def. 1.3.2
- moreover, $a_n \leq x$
- Therefore, $a_n \leq x \leq b_n$

Remark! The NIP requires the intervals to be closed!

The rational number are dense in $\mathbb{R}$

Theorem 0.0.8

$$\forall a, b \in \mathbb{R} \text{ } with \text{ } a < b \text{ } \exists r \in \mathbb{Q} \text{ } s.t \text{ } a < r < b$$

*Proof:*Only case $0 \leq a < b$:

- AP$\Rightarrow$ there exist $n, m \in \mathbb{N}$ such that $1/n < b - a$ and $na < m$
- we can choose this an small enough to be sandwich by $m, m-1 \Rightarrow m-1 \leq na < m$
- $m \leq na + 1 < n(b - \frac{1}{n}) + 1 = nb$
- hence, $m \leq nb$ and $na < m$ which means that $a < \frac{m}{n} < b$

Corollary 0.0.9 (Density of in $\mathbb{R}$) *Given two real numbers $a < b$, there exists an irrational number satisfying $a < t < b$*

Existence of square roots

Theorem 0.0.9 $\exists \alpha \in \mathbb{R}$ $s.t.$ $\alpha^2 = 2$

Cardinality

The term cardinality is used in mathematics to refer to the size of a set.

1-1 Correspondence

Definition 0.0.10 (A *one-to-one* or injective, surjective, bijective functions) *A function $f : A \to B$ is*

- one-to-one *(1-1) if $a_1 = a_2$ in A implies that $f(a_1) = f(a_2)$ in B.*
- onto *or* surjective *if, given any $b \in B$, it is possible to find an element $a \in A$ for which $f(a) = b$.*

- bijective *if f is both injective and surjective*

Definition 0.0.11 *Two sets A, B have the same cardinality if there exists a bijective function $f : A \to B$*
Notation: $A \sim B$

Theorem 0.0.10 $\sim$ *is an equivalence relation:*

(i). $A \sim A$

(ii). $A \sim B \Leftrightarrow B \sim A$

(iii). $A \sim B$ *and* $B \sim C \Rightarrow A \sim C$

Countable sets

Definition 0.0.12 *A set A is called*

- countable *if $A \sim S$ for some $S \subseteq \mathbb{N}$*
- uncountable *otherwise*

Lemma 0.0.5 *A countable $\Leftrightarrow \exists f : A \to \mathbb{N}$ injective*

Lemma 0.0.6 *A countable $\Leftrightarrow \exists g : \mathbb{N} \to A$ surjective*

Corollary 0.0.13

$$\left.\begin{array}{c} B \text{ countable} \\ f : A \to B \text{ injective} \end{array}\right\} \Rightarrow A \text{ countable}$$

$$\left.\begin{array}{cc} A & \text{countable} \\ g : A \to B & \text{surjective} \end{array}\right\} \Rightarrow B \text{ countable}$$

Theorem 0.0.11 *two parts:*

(i). The set $\mathbb{Q}$ is countable

(ii). the set $\mathbb{R}$ is uncountable

Proof(ii): Assume $\mathbb{R}$ is countable.
If $g : \mathbb{N} \to \mathbb{R}$ is surjective, then

$$R = \{x_1, x_2, x_3, x_4, ...\} \quad \text{where} \quad x_n = g(n)$$

To show: $\exists x \in \mathbb{R}$ s.t $x \neq x_n \ \forall n \in \mathbb{N}$
Choose closed and bounded intervals as follows:

$$\begin{array}{rcc} l_1 & \text{such that} & x_1 \notin l_1 \\ l_2 \subseteq l_1 & \text{such that} & x_2 \notin l_2 \\ l_3 \subseteq l_2 & \text{such that} & x_3 \notin l_3 \\ & \vdots & \end{array}$$

NIP $\Rightarrow \exists x \in \mathbb{R}$ s.t. $x \in \cap_{n=1}^{\infty} I_n$. But $x \neq x_n$ for all $n \in \mathbb{N}$ because $x_n \notin I_n$.

Corollary 0.0.14 $\mathbb{Q}^c = \mathbb{R} \backslash \mathbb{Q}$

Proof: We know that $\mathbb{Q}$ is countable
$\mathbb{Q}^c$ countable $\Rightarrow \mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ countable. Contradiction! That is, there are "more" irrationals than rationals

Theorem 0.0.12 *If $A \subseteq B$ and B is countable, then A is either countable or finite*

Theorem 0.0.13 *two parts:*

(i). if $A_1, A_2, ..., A_m$ are each countable sets, then the union $A_1 \cup A_2 \cup \cdots \cup A_m$ countable

(ii). If A_n is countable set for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n$ is countable

Cantor's Theorem

Cantor published his discovery that $\mathbb{R}$ is uncountable in 1874.

Theorem 0.0.14 *The open interval* $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ *is uncountable*

Proof: take any $g : \mathbb{N} \to (0,1)$, then

$$\begin{aligned} g(1) &= 0.d_{11}d_{12}d_{13}d_{14}\cdots \\ g(2) &= 0.d_{21}d_{22}d_{23}d_{24}\cdots \\ g(3) &= 0.d_{31}d_{32}d_{33}d_{34}\cdots \\ &\vdots \end{aligned}$$

Define $t \in (0,1)$ by

$$t = 0.c_1c_2c_3c_4\cdots c_n = \begin{cases} 2 & \text{if } d_{nn} \neq 2 \\ 3 & \text{if } d_{nn} = 2 \end{cases}$$

Then $t \neq g(n)$ for all $n \in \mathbb{N}$ so g is not surjective

Sequences and Series

The limit of a Sequence

Definition 0.0.15 *A sequence is a function whose domain is* $\mathbb{N}$

Definition 0.0.16 (Convergence of a Sequence) a_n *converges to* a *if*

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t. \quad n \geq N \quad \Rightarrow \quad |a_n - a| < \epsilon$$

Notation: $a = \lim a_n$ *or* $(a_n) \to a$

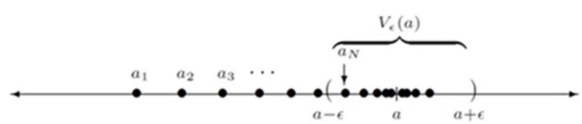
Definition 0.0.17 (neighborhood) *For* $a \in \mathbb{R}$ *and* $\epsilon > 0$ *the set*

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ*-neighborhood of* a

Definition 0.0.18 (Convergence of a sequence: topological version) *A sequence* (a_n) *converges to* a *if, given any* ϵ*-neighborhood* $V_\epsilon(a)$ *of* a*, there exist a point in the sequence after which all of the terms are in* $V_\epsilon(a)$*. In other words, every* ϵ*-neighborhood contains all but a finite number of the terms of* a_n*:*

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ s.t. \ n \geq N \ \Rightarrow \ a_n \in V_\epsilon(a)$$



Moral: the tail of the sequence gets trapped in $V_\epsilon(a)$

Theorem 0.0.15 (Uniqueness of Limits) *The limit of a sequence, when it exists, must be unique*

Standard limits

- $\lim 1/n^\alpha = 0 \quad (\alpha > 0)$
- $\lim c^n = 0 \quad (-1 < c < 1)$
- $\lim c^n n^\alpha = 0 \quad (-1 < c < 1, \alpha \in \mathbb{R})$
- $\lim \sqrt[n]{c} = 1 \quad (c > 0)$
- $\lim \sqrt[n]{n} = 1$
- $\lim n!/n^n = 0$

Definition 0.0.19 (divergent sequence) *A sequence that does not converge is called divergent*

For understand what does it mean we need to obtain a **Logical negation** from the definition of convergence.

Logical negation:

$$\exists \epsilon > 0 \text{ s.t } \forall N \in \mathbb{N} \text{ s.t. } |a_n - a| \geq \epsilon$$

Definition 0.0.20 *(a_n) is bounded if*

$$\exists M > 0 \quad s.t \quad |a_n| \leq M \quad \forall n \in \mathbb{N}$$

Theorem 0.0.16 *if (a_n) is convergent $\Rightarrow$ (a_n) is bounded*

Proof: let $a = \lim a_n$, then for $\epsilon = 1$ there exist $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N &\Rightarrow |a_n - a| < 1 \\ &\Rightarrow ||a_n| - |a|| < 1 \\ &\Rightarrow |a_n| - |a| < 1 \\ &\Rightarrow |a_n| < 1 + |a| \end{aligned}$$

For $M = max\{|a_1|, |a_2|, |a_3|, ..., |a_{N-1}|, 1 + |a|\}$ we have

$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

Warning: the converse is not true!
NOTE: Theorem can be used to prove that a sequence diverges

0.1 Algebraic properties

Theorem 0.1.1 *if $a = \lim a_n$ and $b = \lim b_n$ then*

(i). $\lim(ca_n) = ca$ *where* $c \in \mathbb{R}$

(ii). $\lim(a_n + b_n) = a + b$

(iii). $\lim(a_n b_n) = ab$

(iv). $\lim(a_n/b_n) = a/b$ *if* $b \neq 0$

Proof (ii):

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \end{aligned}$$

Let $\epsilon > 0$ be arbitrary, then

$$\begin{aligned} \exists N_1 \in \mathbb{N} \quad s.t. \quad n \geq N_1 \quad &\Rightarrow |a_n - a| < \frac{1}{2}\epsilon \\ \exists N_2 \in \mathbb{N} \quad s.t. \quad n \geq N_1 \quad &\Rightarrow |a_n - a| < \frac{1}{2}\epsilon \end{aligned}$$

Define $N = max\{N_1, N_2\}$ *then*

$$n \geq N \quad \Rightarrow \quad |(a_n - b_n) - (a + b)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

Proof (iii):

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &= |b_n(a_n - a) + a(b_n - b)| \\ &\leq |b_n(a_n - a)| + |a(b_n - b)| \\ &= |b_n||a_n - a| + |a||b_n - b| \\ &\leq M|a_n - a| + |a||b_n - b| \quad (b_n) \textit{is convergent and therefore bounded} \end{aligned}$$

Let $\epsilon > 0$ *be arbitrary, then*

$$\exists N_1 \in \mathbb{N} \quad s.t. \quad n \geq N_1 \quad \Rightarrow |a_n - a| < \frac{1}{2M}\epsilon$$
$$\exists N_2 \in \mathbb{N} \quad s.t. \quad n \geq N_1 \quad \Rightarrow |b_n - b| < \frac{1}{2|a|}\epsilon$$

Define $N = max\{N_1, N_2\}$ *then*

$$n \geq N \quad \Rightarrow \quad |a_n b_n - ab| < \frac{1}{2M}\epsilon + \frac{1}{2|a|}\epsilon = \epsilon$$

Order properties

Theorem 0.1.2 (order limit theorem) *if* $\lim a_n = a$ *and* $\lim b_n = b$ *then*

(i). $a_n \geq 0 \quad \forall n \in \mathbb{N} \quad \Rightarrow a \geq 0$

(ii). $a_n \leq b_n \quad \forall n \in \mathbb{N} \quad \Rightarrow a \leq b$

(iii). $c \leq b_n \quad \forall n \in \mathbb{N} \quad \Rightarrow c \leq b$

(iv). $a_n \leq c \quad \forall n \in \mathbb{N} \quad \Rightarrow a \leq c$

Proof (i): assume that $a < 0$
For $\epsilon = |a|$ there exist $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N \quad &\Rightarrow |a_n - a| < \epsilon \\ &\Rightarrow -\epsilon < a_n - a < \epsilon \\ &\Rightarrow a - \epsilon < a_n < a + \epsilon \\ &\Rightarrow a_n < a + |a| = 0 \\ &\quad \text{Contradiction!} \end{aligned}$$

Note: Loosely speaking, limits and their properties do not depend at all on what happens at the beginning of the sequence but are *strictly* determined by what happens when n gets large. In the language of analysis, when a property is not necessarily possessed by some finite number of initial terms but is possessed by all terms in the sequence after some point N, we say that the sequence *eventually* has this property.

Theorem 0.1.3 (Squeeze theorem) *If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$ as well.*

Proof: Given $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that whenever $n \geq N_1, n \geq N_2$, $|x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$

Choose $N = max\{N_1, N_2\}$ then we get whenever $n \geq N$, $|x_n - l| < \epsilon, |z_n - l| < \epsilon$. This gives

$$-\epsilon < x_n - l \leq y_n - l \leq z_n - l < \epsilon$$
$$-\epsilon < y_n - l < \epsilon \Rightarrow |y_n - l| < \epsilon$$

or
If $y = \lim y_n$ then by thm $y_n \leq z_n \Rightarrow y \leq l$ and $x_n \leq y_n \Rightarrow l \leq y$. Therefore, $l \leq y \leq l$. Hence, $y = l$.

The monotone convergence theorem and infinite series

Definition 0.1.1 *(a_n) is called monotone if is either*

- *increasing:* $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$
- *decreasing:* $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$

Theorem 0.1.4 (Monotone converges theorem (MCT)) *(a_n) bounded & monotone $\Rightarrow$ (a_n) converges. $a = \lim a_n$ exist*

Proof: $A = \{a_n : n \in \mathbb{N}\}$ is bounded Strategy of proof:

- a_n increasing $\Rightarrow \lim a_n = supA$
- a_n decreasing $\Rightarrow \lim a_n = infA$

Assume that (a_n) increases

Let $s = sup\{a_n : n \in \mathbb{N}\}$

Let $\epsilon > 0$ be arbitrary, then $s - \epsilon$ is not an upper bound. Therefore, there exists $N \in \mathbb{N}$ s.t. $s - \epsilon < a_N$.
For $n \geq N$ we have

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon \quad \Rightarrow |a_n - s| < \epsilon$$

Assume that (a_n) decreses

Let $i = inf\{a_n : n \in \mathbb{N}\}$

Let $\epsilon > 0$ and arbitrary, then $i + \epsilon$ is not an lower bound. Therefore, there exist $N \in \mathbb{N}$ s.t $a_N < i + \epsilon$.
For $n \geq N$ we have

$$i + \epsilon > a_N \geq a_n \geq i > i - \epsilon \quad \Rightarrow |a_n - i| < \epsilon$$

Subsequences

Definition 0.1.2 *pick $n_k \in \mathbb{N}$ such that*

$$1 \leq n_1 < n_2 < n_3 < \cdots$$

If (a_n) is a sequence then

$$(a_{n_k}) = (a_{n_1}, a_{n_2}, a_{n_3,...})$$

is called a subsequence of (a_n). Note: $n_k \geq k$ since $k \in \mathbb{N}$

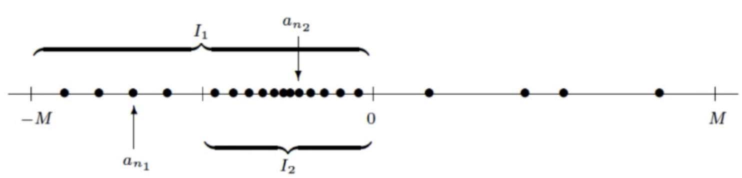
Theorem 0.1.5 $\lim a_n = a \Rightarrow \lim a_{n_k} = a$

Proof: let $\epsilon > 0$ be arbitrary, then

$$\begin{aligned} \exists N \in \mathbb{N} \quad \text{s.t} \quad & n \geq N \Rightarrow |a_n - a| < \epsilon \\ & k \geq N \Rightarrow n_k \geq N \\ & \qquad \Rightarrow |a_{n_k} - a| < \epsilon \end{aligned}$$

Theorem 0.1.6 (Bolzano-Weierstrass theorem) *Every bounded sequence has a convergent subsequence.*

Proof: There exists $M > 0$ such that $a_n \in [-M, M]$ for all n



Bisect the closed interval $[-M, M]$ into two closed intervals $[-M, 0], [0, M]$. Halving-process gives nested closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

NIP $\Rightarrow$ there exists $x \in \bigcap_{n=1}^{\infty} I_n$

each I_k contains infinitely many terms of the seq.

- pick $n_1 \in \mathbb{N}$ with $a_{n_1} \in I_1$

- pick $n_2 \in \mathbb{N}$ with $n_2 > n_1$ and $a_{n_2} \in I_2$
- pick $n_3 \in \mathbb{N}$ with $n_3 > n_1$ and $a_{n_3} \in I_3$

$\vdots$

Note that

$$\left.\begin{array}{ll} x & \in I_k \\ a_{n_k} & \in I_k \end{array}\right\} \Rightarrow |a_{n_k} - x| \leq length(I_k) = \frac{2M}{2^k} \to 0$$

Infinitely series 1

Definition 0.1.3

- ***Infinite series:***

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

- ***n-th partial sum:***

$$s_n = a_1 + a_2 + \cdots + a_n$$

- *if* $\lim s_n = s$, *then we say the series* converges *to* s

Theorem 0.1.7 (Euler's famous example)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \quad \textit{converges}$$

Proof:

$$\begin{aligned} s_n &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \\ s_n &< s_{n+1} \quad \forall n \in \mathbb{N} \\ s_n &< 2 \\ MCT &\Rightarrow \lim s_n \text{ exists} \end{aligned}$$

This because

$$\begin{aligned} s_n &= 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \cdots + \frac{1}{n \cdot n} \\ &< 1 + 12 \cdot 1 + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \cdots + \frac{1}{n \cdot (n-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 - \frac{1}{n} \\ &< 2 \end{aligned}$$

Remark: since $s_n < 2$ for all n the order limit theorem implies

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim s_n \leq 2$$

Euler proved in 1734 that in fact

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Theorem 0.1.8 (harmonic seires)

$$\sum_{k=1}^{\infty} \frac{1}{k} \quad \textit{diverges}$$

The integral test for convergence

Theorem 0.1.9 *assume that* $f : [1, \infty] \to \mathbb{R}$ *is*

(i). positive

(ii). continuous

(iii). monotonically decreasing

Let $a_k = f(k)$ *then*

$$\sum_{k=1}^{\infty} a_k \quad \textit{converges} \Leftrightarrow \int_1^{\infty} f(x)\, dx < \infty$$

The Cauchy Criterion

Definition 0.1.4 (Cauchy sequence) (a_n) *is a Cauchy sequence if*

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t. \quad n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$$

Meaning: *the terms get close to each other*

Theorem 0.1.10 (a_n) *convergent* $\Rightarrow (a_n)$ *Cauchy*

Proof: assume $a = \lim a_n$
For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N \quad &\Rightarrow |a_n - a| < \frac{1}{2}\epsilon \\ m, n \geq N \quad &\Rightarrow |a_n - a_m| = |(a_n - a) - (a_m - a)| \\ &\leq |a_n - a| + |a_m - a| \\ &< \epsilon \end{aligned}$$

Lemma 0.1.1 *(a_n) Cauchy $\Rightarrow (a_n)$ bounded*

Proof: for $\epsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n, m \geq N \quad & \rightarrow |a_n - a_m| < 1 \\ n \geq N \quad & \Rightarrow |a_n - a_N| < 1 \\ & \Rightarrow ||a_n| - |a_N|| < 1 \\ & \Rightarrow |a_n| - |a_N| < 1 \\ & \Rightarrow |a_n| < 1 + |a_N| \end{aligned}$$

For $M = max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |a_N|\}$ we have

$$|a_n| \leq M \quad \text{for all } n \in \mathbb{N}$$

Theorem 0.1.11 (Cauchy Criterion) *(a_n) Cauchy $\Rightarrow (a_n)$ convergent*

Proof:
Lemma $\Rightarrow (a_n)$ is bounded
For weistrass-bolzano $\Rightarrow (a_n)$ has a convergent subsequence (a_{n_k}) $a = \lim a_{n_k}$
For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$n, m \geq N \quad \Rightarrow |a_n - a_m| < \frac{1}{2}\epsilon$$

Fix an index $n_k \geq N$ such that $|a_{n_k} - a| < \frac{1}{2}\epsilon$, then

$$\begin{aligned} n \geq N \quad \Rightarrow |a_n - a| & = |a_n - a_{n_k} + a_{n_k} - a| \\ & \leq |a_n - a_{n_k}| + |a_{n_k} - a| \\ & < \epsilon \end{aligned}$$

Infinite Series Properties

Theorem 0.1.12 (Algebraic Limit Theorem for series) *if $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ then*

(i). $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$

(ii). $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$

Theorem 0.1.13 (Cauchy Criterion) *the following statements are equivalent*

(i). $\sum_{k=1}^{\infty} a_k$ *converges*

(ii). for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ s.t.

$$n > m \geq N \Rightarrow |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

Proof: note that

$$|s_n - s_m| = |a_{m+1} + \cdots + a_n|$$

Statement 1 $\Leftrightarrow$ (s_n) converges $\Leftrightarrow$ (s_n) Cauchy $\Leftrightarrow$ Statement 2

Theorem 0.1.14 $\sum_{k=1}^{\infty} a_k$ *converges* $\Rightarrow \lim a_k = 0$

Proof: let $\epsilon > 0$ be arbitrary
There exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n > m \geq N \quad &\Rightarrow |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon \\ n = m+1 \text{ and } m \geq N \quad &\Rightarrow |a_{m+1}| < \epsilon \end{aligned}$$

Warning: the converse is NOT true!
Note: the previous theorem also gives a test for divergence

Theorem 0.1.15 (Comparison test) *if* $0 \leq a_k \leq b_k$ *for all* $k \in \mathbb{N}$*, then*

(i). $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges*

(ii). $\sum_{k=1}^{\infty} a_k$ *diverges* $\Rightarrow \sum_{k=1}^{\infty} b_k$ *diverges*

Proof:

$$\begin{aligned} |a_{m+1} + a_{m+2} + \cdots + a_n| &= a_{m+1} + a_{m+2} + \cdots + a_n \\ &\leq b_{m+1} + b_{m+2} + \cdots + b_n \\ &= |b_{m+1} + b_{m+2} + \cdots + b_n| \end{aligned}$$

Apply the Cauchy criterion for series.
Note: this theorem does not be true for all k, but its sufficient that is true for a k sufficiently large

Theorem 0.1.16 (Alternating series test) *assume*

(i). $0 \leq a_{k+1} \leq a_k$ *for all* $k \in \mathbb{N}$

(ii). $\lim a_k = 0$

then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges*

Proof: consider the partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$$

the partial sums form nested intervals:

$$I_n = [s_{2n}, s_{2n-1}] \quad \Rightarrow I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

NIP $\Rightarrow$ there exists $s \in \mathbb{N}$ such that $s \in I_n$ for all $n \in \mathbb{N}$

Let $\epsilon > 0$ be arbitrary
Choose $N \in \mathbb{N}$ such that $a_{2N} < \epsilon$, then

$$\begin{aligned} n \geq 2N \quad &\Rightarrow s, s_n \in I_N = [s_{2N}, s_{2N-1}] \\ &\Rightarrow |s - s_n| \leq s_{2N-1} - s_{2N} \\ &\Rightarrow |s - s_n| \leq a_{2N} \\ &\Rightarrow |s - s_n| < \epsilon \end{aligned}$$

Theorem 0.1.17 (Absolute vs. conditional convergence) *$\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges*

Proof: note that

$$0 \leq a_k + |a_k| \leq 2|a_k| \quad \text{for all } k \in \mathbb{N}$$

Comparison Test $\Rightarrow \sum_{k=1}^{\infty}(a_k + |a_k|)$ converges
Apply Algebraic Limit Theorem:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k| \quad \text{converges}$$

Definition 0.1.5 *$\sum_{k=1}^{\infty} a_k$ is called*

*(i). **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ converges*

*(ii). **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |a_k|$ diverges*

Definition 0.1.6 (geometric series) *a geometric series is of the form*

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$
$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If and only if $|r| < 1$

Definition 0.1.7 *telescoping series are the form*

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(b_k - b_{k+1})$$

Successive terms cancel each other:

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_n - b_{n+1} \\ &= b_1 - b_{n+1} \end{aligned}$$

The series converges $\Leftrightarrow (b_n)$ converges

Basic Topology of R

Interval

Definition 0.1.8 *Closed interval (endpoints included):*

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

Definition 0.1.9 *Open interval (endpoints not included):*

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

Definition 0.1.10 *$O \subset \mathbb{R}$ is open if*

$$\forall a \in O \quad \exists \epsilon > 0 \quad s.t. \quad V_\epsilon(a) \subset O$$

Note: *the empty set $\emptyset$ is open by definition*

Theorem 0.1.18

*(i). Unions of **arbitrary** collections of open sets are open*

*(ii). Intersections of **finite** collections of open sets are open*

Proof(i): let $O = \bigcup_{i \in I} O_i$ with each O_i open
$x \in O \Rightarrow x \in O_i$ for some $i \in I$
There exists $\epsilon > 0$ such that $V_\epsilon(x) \subset O_i \subset O$

Proof(ii): let $O = O_1 \cap O_2 \cap \cdots \cap O_n$ with each O_i open
$x \in O \Rightarrow x \in O_i$ for all $i = 1, ..., n$
For all $i = 1, ..., n$ there exists $\epsilon_i > 0$ such that $V_{\epsilon_i}(x) \subset O_i$ For $\epsilon = min\{\epsilon_1, ..., \epsilon_n\}$
we have $V_\epsilon(x) \subset O_i$ for all $i = 1, ..., n$

Warning: the intersection of infinitely many open sets need not be open!

Definition 0.1.11 (limit point) *x is a limit point of $A \subset \mathbb{R}$ if $\forall \epsilon > 0$ $V_\epsilon(x)$ intersects A in some point other than x*

Note: Limit points of A may or may not belong to A

Theorem 0.1.19 *The following statements are equivalent:*

(i). x is a limit point of A

(ii). There exists a sequence a_n in A such that

$$a_n \neq x \quad \forall n \in \mathbb{N} \quad and \quad x = \lim a_n$$

Proof (i,ii): let $n \in \mathbb{N}$ and set $\epsilon = 1/n$

There exists $a_n \in V_\epsilon(x) \cap A$ with $a_n \neq x$

Note that $|a_n - x| < \epsilon = \frac{1}{n}$
Proof (ii,i): for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |a_n - x| < \epsilon$$

In particular, $a_N \in V_\epsilon(x)$
By assumption $a_N \neq x$ and $a_N \in A$

Definition 0.1.12 (Closed set) *A set is closed if it contains its limit points*

Theorem 0.1.20 *the following statements are equivalent*

(i). F is closed

(ii). Every Cauchy sequence in F has its limit in F

Proof (i,ii): Let $(a_n) \subset F$ be Cauchy

$x = \lim a_n$ exists; now consider two cases:

- $x \neq a_n$ for all $n \in \mathbb{N} \Rightarrow x$ is a limit point of $F \Rightarrow x \in F$
- $x = a_n$ for some $n \in \mathbb{N} \Rightarrow x \in F$ holds trivally

Proof(ii,i): let x be a limit point of F

$x = \lim a_n$ with $a_n \in F$ and $a_n \neq x$ for all $n \in \mathbb{N}$

(a_n) is convergent $\Rightarrow (a_n)$ Cauchy $\Rightarrow x \in F$ by assumption

Definition 0.1.13 (Closure) *the closure of A is defined as*

$$\bar{A} = A \cup \{\textit{all limit points of } A\}$$

Theorem 0.1.21 $\bar{A}$ *is closed*

Proof: show that x limit point of $\bar{A} \Leftrightarrow x$ limit point of A

$\bar{A} = A \cup L$ with $L = \{\text{limit points of } A\}$

x limit point of $\bar{A} \Rightarrow \forall \epsilon > 0 \quad \exists y \in V_\epsilon(x) \cap \bar{A} \quad y \neq x$
Note: either $y \in A$ or $y \in L$

(i). $y \in A \Rightarrow x$ is a limit point of A

(ii). $y \in L \Rightarrow \forall \delta > 0 \quad \exists z \in V_\delta(y) \cap A \quad z \neq y$

Note: $V_\delta(y) \subset V_\epsilon(x)\backslash\{x\}$ for δ small enough

Therefore x is a limit point of A

Theorem 0.1.22 (complements)

(i). O open $\Leftrightarrow O^c$ closed

(ii). F closed $\Leftrightarrow F^c$ open

Warning: sets are not likes doors!

- $(0, 1]$ and $\mathbb{Q}$ are neither open nor closed
- $\mathbb{R}$ and $\emptyset$ are both open and closed

Practical consequence: it is impossible to prove openness/ closedness by contradiction

Theorem 0.1.23 (unions and intersections)

(i). Unions of finite collections of closed sets are closed

(ii). Intersections of arbitrary collections of closed sets are closed

Proof(i):

$$\begin{aligned} F_1, ..., F_n \text{ closed} &\Rightarrow F_1^c, ..., F_n^c \text{ open} \\ &\Rightarrow F_1^c \cap \cdots \cap F_n^c \text{ open} \\ &\Rightarrow (F_1^c \cap \cdots \cap F_n^c)^c \text{ closed} \\ &\Rightarrow F_1 \cup \cdots \cup F_n \text{ closed} \end{aligned}$$

Proof (ii):

$$\begin{aligned} F_i \text{ closed for all } i \in I &\Rightarrow F_i^c \text{ open for all } i \in I \\ &\Rightarrow \bigcup_{i \in I} F_i^c \text{ open} \\ &\Rightarrow (\bigcup_{i \in I} F_i^c)^c \text{ closed} \\ &\Rightarrow \bigcup_{i \in I} F_i \text{ closed} \end{aligned}$$

The last passage of both proof we have used De Morgan's laws, which state that for any collection of sets $\{E_i : i \in I\}$

$$\left(\bigcup_{i \in I} E_i\right)^c = \bigcap_{i \in I} E_i^c \quad \text{and} \quad \left(\bigcap_{i \in I} E_i\right)^c = \bigcup_{i \in I} E_i^c$$

Warning: the union of infinitely many closed sets need not be closed

Compact sets

Definition 0.1.14 (sequential definition) *$K \subset \mathbb{R}$ is compact if* every *sequence in K has a convergent subq. with a limit in K*

Theorem 0.1.24 *$K \subset \mathbb{R}$ compact $\Leftrightarrow$ K closed and bounded*

Proof($\Rightarrow$): Assume K is not bounded. There exists $(x_n) \subset K$ with $|x_n| > n$ for all $n \in \mathbb{N}$.

(x_n) has no convergent subsequence. Contradiction!

Let x be a limit point of K. There exists $(x_n) \subset K$ such that $x = \lim x_n$.

K compact $\Rightarrow$ there exists a subsequence $(x_{n_k}) \to y \in K$. $(x_{n_k}) \to x$ as well $\Rightarrow x = y \in K$

Proof($\Leftarrow$): let $(x_n) \subset K$. K is bounded $\Rightarrow$ (x_n) is bounded.

B-W Theorem $\Rightarrow$ (x_n) has a convergent subsequence. Let $x = \lim x_{n_k}$. Hence, K is closed $\Rightarrow x \in K$

Theorem 0.1.25 (Generalization of the NIP) *assume that $K_n \neq \emptyset$ is compact for all $n \in \mathbb{N}$ and*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$$

Then $\bigcap_{n=1}^{\infty} K_n$ is nonempty

Open covers

Definition 0.1.15 *Let $A \subset \mathbb{R}$ and assume that the sets $O_i \subset \mathbb{R}$ where $i \in I$, are open. We call the sets O_i an open cover for A if*

$$A \subset \bigcup_{i \in I} O_i$$

Theorem 0.1.26 *K compact $\Leftrightarrow$ any open cover for K has a finite subcover*

Proof($\Rightarrow$):

Let $O_i, i \in I$, be an open cover for K without finite subcover.

Take a bounded, closed interval $J_1 \supset K$

Halving process: construct J_n be closed intervals s.t.

- $J_1 \supset J_2 \supset J_3 \supset \cdots$
- $K \cap J_n$ can not be coverd by finitely many O_i's

$K \cap J_n$ compact for all $n \in \mathbb{N} \Rightarrow \bigcap_{n=1}^{\infty}(K \cap J_n) \neq \emptyset$.

There exists $x \in K$ such that $x \in J_n$ for all n

$x \in O_i$ for some $i \in I$ and let $\epsilon > 0$ such that $V_\epsilon(x) \subset O_i$

There exists $N \in \mathbb{N}$ such that length$(J_N) < \epsilon$

Hence, $K \cap J_N \subset J_N \subset V_\epsilon(x) \subset O_i$. Contradiction!

Proof($\Leftarrow$):

$O_n = (-n, n), n \in \mathbb{N}$, is an open cover for K.

$K \subset O_1 \cup O_2 \cup \cdots \cup O_N = (-N, N)$ for some $N \in \mathbb{N}$. Therefore, K is bounded.

Let y be a limit point K

There exists $(y_n) \subset K$ with $y = \lim y_n$. Assume $y \notin K$

Let $x \in K$ and $O_x = V_\epsilon(x)$ with $\epsilon = \frac{1}{2}|x - y|$

The sets, O_x, where $x \in K$, form an open cover for K

There exist $x_1, ..., x_n \in K$ such that $K \subset O_{x_1} \cup \cdots \cup O_{x_n}$

Pick $N \in \mathbb{N}$ such that $|y_N - y| < min\{\frac{1}{2}|x_i - y| : i = 1, ..., n\}$

Hence, $y_N \notin O_{x_1} \cup \cdots \cup O_{x_n}$ Contradiction!

Theorem 0.1.27 (Heine-Borel) *Let $K \subset \mathbb{R}$, the following statements are equivalent:*

(i). K is compact

(ii). K is closed and bounded

(iii). Any open cover for K has a finite sets.

Functional Limits and Continuity

Definition 0.1.16 *Let $f : A \to \mathbb{R}$ and c a limit point of A. We say that $\lim_{x \to c} f(x) = L$ when*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t \begin{Bmatrix} 0 < |x - c| < \delta \\ x \in A \end{Bmatrix} \Rightarrow |f(x) - L| < \epsilon$$

Note: f need not be defined at c

Theorem 0.1.28 (Sequential characterization) *Let $f : A \to \mathbb{R}$ and c a limit point of A.*
The following statements are equivalent

(i). $\lim_{x\to c} f(x) = L$

(ii). $\lim f(x_n) = L$ *for all* $(x_n) \subset A$ *with* $x_n \neq c$ *and* $\lim x_n = c$

Corollary 0.1.17 *consider $f : A \to \mathbb{R}$ and let c be a limit point of A.* $\lim_{x\to c} f(x)$ *does not exist if there exist x_n,$y_n \subset A$ s.t.*

- $x_n \neq c$ *and* $y_n \neq c$
- $\lim x_n = \lim y_n = c$
- $\lim f(x_n) \neq \lim f(y_n)$

Theorem 0.1.29 (Algebraic properties) *Let $f : A \to \mathbb{R}$, c a limit point of A, and*

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = M$$

Then

(i). $\lim_{x\to c} kf(x) = kl \;\; k \in \mathbb{R}$

(ii). $\lim_{x\to c}[f(x) + g(x)] = L + M$

(iii). $\lim_{x\to c}[f(x)g(x)] = LM$

(iv). $\lim_{x\to c}[f(x)/g(x)] = L/M$ *provided* $M \neq 0$

Definition 0.1.18 *$f : A \to \mathbb{R}$ is continuous at $c \in A$ if*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad s.t \left\{ \begin{array}{c} |x - c| < \delta \\ x \in A \end{array} \right\} \Rightarrow |f(x) - f(c)| < \epsilon$$

Notes: $f(c)$ needs to be defined, but c need not be a limit point of A. Moreover, δ may depend on both ϵ and c

Example: if $c \in A$ is isolated then $f : A \to \mathbb{R}$ is continuous at c.

Let $\epsilon > 0$ be arbitrary

Take $\delta > 0$ such that $V_\delta(c) \cap A = \{c\}$, then

$$\begin{aligned} |x - c| < \delta \quad \text{and } x \in A &\Rightarrow x \in V_\delta(c) \cap A \\ &\Rightarrow x = c \\ &\Rightarrow f(x) = f(c) \\ &\Rightarrow |f(x) - f(c)| = 0 < \epsilon \end{aligned}$$

Theorem 0.1.30 *let $f : A \to \mathbb{R}$ and $c \in A$. the following statements are equivalent:*

(i). f *is continuous at* c

(ii). $(x_n) \subset A$ *and* $\lim x_n = c \Rightarrow \lim f(x_n) = f(c)$

If c is a limit point of A then (i) and (ii) are also equivalent with

(iii). $\lim_{x\to c} f(x) = f(c)$

Corollary 0.1.19 *let $f : A \to \mathbb{R}$ and $c \in A$ a limit point, f is not continuous at $x = c$ if there exists $(x_n) \subset A$ s.t*

- $x \neq c$
- $\lim x_n = c$
- $\lim f(x_n) \neq f(c)$

Continuity and compactness

Theorem 0.1.31 *$f : A \to \mathbb{R}$ cont. and $K \subset A$ compact $\Rightarrow f(K)$ compact*

Proof: Let $(y_n) \subset f(K)$ be arbitrary

There exists $(x_n) \subset K$ such that $y_n = f(x_n)$ for all n

K compact $\Rightarrow$ some subsequence $x_{n_k} \to x \in K$

f continuous $\Rightarrow y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$

Warning: the previous theorem is false for pre-image:

$$f^{-1}(K) = \{x \in A : f(x) \in K\}$$

Theorem 0.1.32 (Maxima and Minima) *Let $K \subset \mathbb{R}$ be compact and $f : K \to \mathbb{R}$ continuous, then f attains a maximum and a minimum on K*

Proof (max): $f(K)$ is compact

$s = sup f(K)$ exists and $s \in f(K)$

$s = f(c)$ for some $c \in K$

s is an upper bound for $f(K) \Rightarrow f(x) \leq s$ for all $x \in K$

Warning: without compactness the previous theorem is false!

Uniform continuity

Theorem 0.1.33 *$f : A \to \mathbb{R}$ is uniformly continuous on A if*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \textit{such that } \forall x, y \in A \quad |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Note: uniform means that δ does not depend on x or y

Logical negation: $\exists \epsilon_0 > 0$ such that $\forall \delta > 0$ $\exists x, y \in A$ for which

$$|x-y| < \delta \quad \text{but} \quad |f(x) - f(y)| \geq \epsilon_0$$

Theorem 0.1.34 *the following statements are equivalent*

(i). *$f : A \to \mathbb{R}$ is not uniformly continuous on A*

(ii). *There exists $\epsilon_0 > 0$ and $(x_n), (y_n) \subset A$ such that*

$$|x_n - y_n| \to 0 \quad \textit{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0 \quad \textit{for all } n$$

Theorem 0.1.35 *if $f : K \to \mathbb{R}$ is continuous and K is compact then f is uniformly continuous on K*

Proof: let $\epsilon > 0$ be arbitrary

For all $c \in K$ there exists $\delta_c > 0$ such that

$$|x - c| < 2\delta_c \quad \Rightarrow \quad |f(x) - f(c)| < \frac{1}{2}\epsilon \quad \text{for cosmetic purposes}$$

$O_c = (c - \delta_c, c + \delta_c)$, with $c \in K$, form an open cover for K

$K \subset O_{c_1} \cup \cdots \cup O_{c_n}$ for some $c_1, ..., c_n \in K$

Take $x, y \in K$ with $|x - y| < \delta = min\{\delta_{c_1}, ..., \delta_{c_n}\}$

(1)

$$|x - c_i| < \delta_{c_i} \quad \text{for some } i = 1, ..., n$$
$$|f(x) - f(y)| < \frac{1}{2}\epsilon$$

(2)

$$|c_i - y| \leq |c_i - x| + |x - y| < \delta_{c_i} + \delta \leq 2\delta_{c_i}$$
$$|f(c_i) - f(y)| < \frac{1}{2}\epsilon$$

Apply triangle inequality with the (1) and (2) we have proved that the theorem holds.

Intermediate value theorem

Theorem 0.1.36 *if* $f : [a, b] \to \mathbb{R}$ *is continuous and*

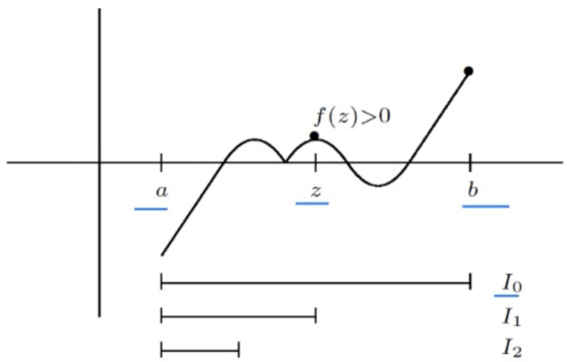
$$f(a) < L < f(b) \quad or \quad f(a) > L > f(b)$$

then $f(c) = L$ *for some* $c \in (a, b)$

Proof: without loss of generality we can assume

- $L = 0$, otherwise replace $f(x)$ by $f(x) - L$
- $f(a) < 0 < f(b)$, otherwise replace $f(x)$ by $-f(x)$

the bisection method gives nested intervals I_n:



At the left endpoint of each I_n we have $f < 0$

At the right endpoint of each I_n we have $f \geq 0$

there exist intervals $I_n = [a_n, b_n]$ such that

- $f(a_n) < 0$ and $f(b_n) \geq 0$
- $I_0 \supset I_1 \supset I_2 \supset \cdots$
- $length(I_n) = (b - a)/2^n$

NIP$\Rightarrow$ $\exists c \in [a, b]$ such that $c \in I_n = [a_n, b_n]$ $\forall$

Derivatives

Definition 0.1.20 *Let* $I \subseteq \mathbb{R}$ *be an interval and* $f : I \to \mathbb{R}$, f *is called differentiable at* $c \in I$ *if*

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad exists$$

Theorem 0.1.37 $f : I \to \mathbb{R}$ *differentiable at* $c \in I \Rightarrow f$ *continuous at* c

Proof:

$$\begin{aligned}\lim_{x\to c}[f(x)-f(c)] &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c}(x-c)\\ &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}[x-c]\\ &= f'(c)\cdot 0\\ &= 0\end{aligned}$$

Theorem 0.1.38 (Interior extremum theorem) *assume*

- $f:(a,b)\to\mathbb{R}$ *is differentiable*
- f *attains a maximum or minimum at* $c\in(a,b)$

then $f'(c)=0$

Proof (maximum): $f(c)\geq f(x)$ for all $x\in(a,b)$

Take sequences (x_n) and (y_n) in (a,b) such that

$$x_n<c<y_n \quad \forall n\in\mathbb{N} \quad \text{and} \quad \lim x_n=\lim y_n=c$$

$f'(c)=0$ by the order limit theorem:

$$\begin{aligned}f'(c)&=\lim\frac{f(x_n)-f(c)}{x_n-c}\geq 0\\ f'(c)&=\lim\frac{f(y_n)-f(c)}{y_n-c}\leq 0\end{aligned}$$

Warning: for closed intervals the previous theorem may be false!

Theorem 0.1.39 (Darboux's theorem) *if* $f:[a,b]\to\mathbb{R}$ *is differentiable and*

$$f'(a)<L<f'(b) \quad \text{or} \quad f'(a)>L>f'(b)$$

then there exist $c\in(a,b)$ *with* $f'(c)=L$

Note:

- proof$\neq$ intermediate value theorem applied to f'
- we do not assume f' to be continuous

Proof: restrict to the case $f'(a)<0<f'(b)$, Otherwise replace $f(x)$ by $\pm(f(x)-Lx)$.

claim: $\exists s\in(a,b)$ s.t. $f(s)<f(a)$

Otherwise $f(x)\geq f(a)$ $\forall x\in(a,b)$ so that

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\geq 0 \text{ Contradiction!}$$

Similarly: $\exists t \in (a,b)$ such that $f(t) < f(b)$

$[a,b]$ compact and f continuous $\Rightarrow f$ attains a minimum on $[a,b]$

$f(s) < f(a)$ and $f(t) < f(b) \Rightarrow f$ attains a minimum in (a,b)

Interior extremum theorem $\Rightarrow f$

Mean value theorem

Theorem 0.1.40 (Rolle's theorem) *assume that*

- $f : [a,b] \to \mathbb{R}$ *is continuous and differentiable on* $(a,b$
- $f(a) = f(b)$

then there exists $c \in (a,b)$ *such that* $f'(c) = 0$

Proof: f cont. and $[a,b]$ cpt. $\Rightarrow f$ attains max/min values

$$\begin{aligned} f(a) = f(b) \quad \text{both max and min } &\Rightarrow f \text{ is constant} \\ &\Rightarrow f'(x) = 0 \text{ for all } x \\ &\Rightarrow \text{ take any } c \in (a,b) \end{aligned}$$

Otherwise, a max or min is attained at $c \in (a,b)$

Then $f'(c) = 0$ by interior extremum theorem

Theorem 0.1.41 (Mean value theorem) *if*

- $f : [a,b] \to \mathbb{R}$ *is continuous*
- f *is differentiable on* (a,b)

Then there exists $c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof: apply Rolle's theorem to

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

then

$$\begin{aligned} &k(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \\ &h(x) = f(x) - k(x) \quad \text{ is continuous on } [a,b] \text{ and differentiable on } (a,b) \\ &h(a) = h(b) = 0 \end{aligned}$$

By Rolle's theorem: $\exists c \in (a,b)$ s.t.

$$h$$

Sequence and Series of Functions

Pointwise convergence

Definition 0.1.21 ***converges pointwise*** *consider* $f_n : A \to \mathbb{R}$

(f_n) *converges pointwise to* $f : A \to \mathbb{R}$ *if for all fixed* $x \in A$

$$\lim f_n(x) = f(x)$$

Thus: for each fixed $x \in A$ we have

$$\forall \epsilon > 0 \quad \exists N_{\epsilon,x} \in \mathbb{N} \quad \text{s.t} \quad n \geq N_{\epsilon,x} \Rightarrow \quad |f_n(x) - f(x)| < \epsilon$$

Uniform convergence

Definition 0.1.22 ***Uniform convergence*** (f_n) *converges uniformly to* $f : A \to \mathbb{R}$ *if*

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad s.t \quad n \geq N_\epsilon \Rightarrow \quad |f_n(x) - f(x)| < \epsilon \quad \forall x \in A$$

Note: uniform means that N_ϵ is independent of $x \in A$

Theorem 0.1.42 *consider* $f_n : A \to \mathbb{R}$ *then*

$$f_n \to f \ \textit{uniformly} \quad \Leftrightarrow \quad \lim \left(\sup_{x \in A} |f_n(x) - f(x)| \right) = 0$$

Proof(⇒): for $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_\epsilon \quad &\Rightarrow \quad |f_n(x) - f(x)| < \epsilon \quad \forall x \in A \\ &\Rightarrow \quad \sup_{x \in A} |f_n(x) - f(x)| \leq \epsilon \end{aligned}$$

Proof(⇐): for $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_\epsilon \quad &\Rightarrow \quad \sup_{x \in A} |f_n(x) - f(x)| < \epsilon \\ &\Rightarrow \quad |f_n(x) - f(x)| < \epsilon \quad \forall x \in A \end{aligned}$$

Theorem 0.1.43 ***Preservation of continuity*** *assume* $f_n : A \to \mathbb{R}$ *satisfies*

(i). $f_n \to f$ *uniformly on* A

(ii). f_n *is continuous at* $c \in A$ *for all* $n \in \mathbb{N}$

Then f *is continuous at* c

Moral: uniform convergence preserves continuity

Proof: for $\epsilon > 0$ there exist

- $N \in \mathbb{N}$ s.t. $|f_N(x) - f(x)| < \frac{1}{3}\epsilon$ for all $x \in A$
- $\delta > 0$ s.t $|x - c| < \delta \Rightarrow |f_N(x) - f_N(c)| < \frac{1}{3}\epsilon$

if $|x - c| < \delta$ then

$$\begin{aligned}|f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon \\ &= \epsilon\end{aligned}$$

Theorem 0.1.44 ***Term-by-term Continuity Theorem*** *Let f_n be continuous functions defined on a set $A \subseteq \mathbb{R}$, and assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f. Then, f is continuous on A*

Theorem 0.1.45 ***Term-by-term Differentiability*** *Let f_n be differentiable functions defined on an interval A, and assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to a limit $g(x)$ on A. If there exists a point $x_0 \in [a, b]$ where $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ satisfying $f'(x) = g(x)$ on A. In other words,*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \textit{and} \quad \sum_{n=1}^{\infty} f_n'(x)$$

Theorem 0.1.46 ***Weierstrass M-test*** *For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying*

$$|f_n(x)| \leq M_n$$

For all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A

Power Series

General form of PS:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Theorem 0.1.47

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges at } c \neq 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} |a_n x^n| \text{ converges for } |x| < |c|$$

Proof:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n c^n \text{ converges} \quad &\Rightarrow \quad \lim a_n c^n = 0 \\ &\Rightarrow (a_n c^n) \text{ is bounded} \\ &\Rightarrow \quad \exists M > 0 \text{ s.t } |a_n c^n| \leq M \ \forall n \in \mathbb{N}\end{aligned}$$

thus,

$$|a_n x^n| = |a_n \left(c \cdot \frac{x}{c}\right)^n| = |a_n c^n| \cdot \left|\frac{x}{c}\right|^n \leq M \cdot \left|\frac{x}{c}\right|^n \qquad \forall n \in \mathbb{N}$$

Note: $|x| < |c| \Rightarrow \left|\frac{x}{c}\right| < 1$

Apply comparison test

$$\sum_{n=0}^{\infty} M \left|\frac{x}{c}\right|^n \quad \text{converges} \quad \Rightarrow \sum_{n=0}^{\infty} |a_n x^n| \quad \text{converges}$$

Corollary 0.1.23 ***Radius of convergence*** *There exists $R \geq 0$ such that*

- $|x| < R \Rightarrow PS$ *converges at* x
- $|x| > R \Rightarrow PS$ *diverges at* x

R is called the ***radius of convergence***

Methods for **computing** R from the a_n's

Root test: if $L = \lim \sqrt[n]{|a_n|}$ exists, then $R = 1/L$

Ratio test: if $L = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $R = 1/L$

If $L = 0$ then $R = \infty$, that is converges on entire real line.

Proof Root Test: $\lim \sqrt[n]{|a_n x^n|} = L|x| \; \forall x \in \mathbb{R}$ fixed

For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ s.t.

$$\begin{aligned} n \geq N \quad &\Rightarrow \quad \left|\sqrt[n]{|a_n x^n|} - L|x|\right| < \epsilon \\ &\Rightarrow \quad L|x| - \epsilon < \sqrt[n]{|a_n x^n|} < L|x| + \epsilon \\ &\Rightarrow \quad (L|x| - \epsilon)^n < |a_n x^n| < (L|x| + \epsilon)^n \end{aligned}$$

thus if $|x| < 1/L$, then pick $\epsilon < 1 - L|x|$

Apply comparison test:

$$\begin{aligned} L|x| + \epsilon < 1 \quad &\Rightarrow \quad \sum_{n=0}^{\infty} (L|x| + \epsilon)^n \text{ converges} \\ &\Rightarrow \quad \sum_{n=0}^{\infty} |a_n x^n| \text{ converges} \\ &\Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n \text{ converges} \end{aligned}$$

instead, if $|x| > 1/L$ then pick $\epsilon < L|x| - 1$

$$\begin{aligned} L|x| - \epsilon > 1 \quad &\Rightarrow \quad (L|x| - \epsilon)^n \text{ unbounded} \\ &\Rightarrow \quad |a_n x^n| \text{ unbounded} \\ &\Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n \text{ diverges} \end{aligned}$$

So far we have discuss only pointwise converge of a power series. Hence, now we will look at uniform convergence

Theorem 0.1.48 ***Uniform convergence***

$$\sum_{n=0}^{\infty} |a_n c^n| \textit{ converges} \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n \textit{ uniformly conv. on } [-|c|, |c|]$$

Proof: for $|x| \leq |c|$ we have

$$|a_n x^n| = |a| \cdot |x|^n \leq |a_n| \cdot |c|^n = |a_n c^n| =: M_n$$

Apply Weierstrass'test:

$$\sum_{n=0}^{\infty} M_n \quad \text{conv.} \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n \quad \text{unif. conv. on } [-|c|, |c|]$$

Corollary 0.1.24 ***Continuity of the limit*** *$\sum_{n=0}^{\infty} a_n x^n$ is continuous function on $(-R, R)$*

Proof: take $x_0 \in (-R, R)$ and $|x_0| < c < d < R$, then

$$\begin{aligned} \text{PS convergent at } d \quad &\Rightarrow \quad \text{PS absolutely convergent at } c \\ &\Rightarrow \quad \text{PS uniformly convergent on } [-c, c] \\ &\Rightarrow \quad \text{PS continuous on } [-c, c] \quad \text{each } a_n x^n \text{ is continuous} \\ &\Rightarrow \quad \text{PS continuous at } x_0 \end{aligned}$$

Corollary 0.1.25

$$\sum_{n=0}^{\infty} |a_n R^n| \textit{ convergent} \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n \textit{ uniformly conv. on } [-R, R]$$

In particular, the PS is continuous on $[-R, R]$

What if convergence is conditional at $x = R$ or $x = -R$?

Lemma 0.1.2 ***Summation by parts*** *if* $s_n = u_1 + \cdots + u_n$*, then*

$$\sum_{k=1}^{n} u_k v_k = s_n v_{n+1} + \sum_{k=1}^{n} s_k(v_k - v_{k+1})$$

Proof: set $s_0 = 0$, then

$$\begin{aligned} u_k v_k &= (s_k - s_{k-1})v_k \\ &= s_k(v_k - v_{k+1}) + s_k v_{k+1} - s_{k-1}v_k \qquad \forall k = 1, ..., n \end{aligned}$$

Lemma 0.1.3 ***Abel's lemma*** *assume that* (u_n) *and* (v_n) *satisfy*

- $|u_1 + \cdots + u_n| \leq C \ \forall n \in \mathbb{N}$
- $0 \leq v_{n+1} \leq v_n \ \forall n \in \mathbb{N}$

Then

$$\left|\sum_{k=1}^{n} u_k v_k\right| \leq C v_1$$

Proof: if $s_n = u_1 + \cdots + u_n$, then

$$\begin{aligned} \left|\sum_{k=1}^{n} u_k v_k\right| &= \left|s_n v_{n+1} + \sum_{k=1}^{n} s_k(v_k - v_{k+1})\right| \\ &\leq |s_n| v_{n+1} + \sum_{k=1}^{n} |s_k|(v_k - v_{k+1}) \\ &\leq C\left(v_{n+1} + \sum_{k=1}^{n}(v_k - v_{k+1})\right) \\ &= C v_1 \end{aligned}$$

Theorem 0.1.49 ***Abel's theorem***

(i). PS converges at $x = R \Rightarrow$ *PS conv. uniformly on* $[0, R]$

(ii). PS converges at $x = -R \Rightarrow$ *PS conv. uniformly on* $[-R, 0]$

Proof(1): for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ s.t

$$n > m \geq N \quad \Rightarrow \quad \left|\sum_{k=m+1}^{n} a_k R^k\right| < \epsilon$$

Take any $x \in [0, R]$ and set

$$v_k = \left(\frac{x}{R}\right)^k, \qquad u_k = \begin{cases} a_k R^k & \text{if } k \geq m+1 \\ 0 & \text{Otherwise} \end{cases}$$

From Abel's lemma we get the Cauchy criterion:

$$\left|\sum_{k=m+1}^{n} a_k x^k\right| = \left|\sum_{k=1}^{n} u_k v_k\right| < \epsilon \cdot \frac{x}{R} \leq \epsilon \qquad \forall x \in [0, R]$$

Theorem 0.1.50 ***Term-wise Differentiability Theorem***

$$\sum_{n=0}^{\infty} a_n x^n \;\; conv. \;\; on \; (-R, R) \quad \Rightarrow \quad \sum_{n=0}^{\infty} n a_n x^{n-1} \;\; conv. \;\; on \; (-R, R)$$

Proof: if $|c| < 1$, then there exists $M > 0$ s.t

$$|nc^{c-1}| \leq M \quad \forall n \in \mathbb{N}$$

Let $|x| < t < R$, then

$$|na_n x^{n-1}| = \frac{1}{t}\left(n\left|\frac{x}{t}\right|^{n-1}\right)|a_n t^n| \leq \frac{M}{t}|a_n t^n|$$

Apply comparison test

Theorem 0.1.51 *For any PS with radius R we have*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \forall x \in (-R, R)$$

Proof: let $0 \leq c < R$, then

- $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges uniformly on $[-c, c]$
- $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 0$

Now apply Term-wise Differentiability Theorem

Taylor Series

Assume f is inf. often differentiable on interval around $x = 0$

Definition 0.1.26 *The Taylor series of f around $x = 0$ is given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 0.1.27

$$s_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \qquad \textit{partial sum}$$
$$E_n(x) = f(x) - s_n(x) \qquad \textit{remainder}$$

Lemma 0.1.4 *assume that*

- $x > 0$ *and* $h(t)$ *is* $n+1$ *times diff.ble on* $[0, x]$
- $h(x) = 0$ *and* $h^{(k)}(0) = 0$ *for all* $k = 0, ..., n$

Then $h^{(n+1)}(c) = 0$ *for some* $c \in (0, x)$

Proof: repeated application of Rolles's theorem gives

$$\begin{aligned} h(0) = h(x) \quad &\Rightarrow \quad h'(c_1) = 0 \ \text{ for some } c_1 \in (0, x) \\ h'(0) = h'(c_1) \quad &\Rightarrow \quad h"(c_2) = 0 \ \text{ for some } c_2 \in (0, c_1) \\ &\vdots \\ h^{(n)}(0) = h^{(n)}(c_n) \quad &\Rightarrow \quad h^{(n+1)}(c_{n+1}) = 0 \ \text{ for some } c_{n+1} \in (0, c_n) \end{aligned}$$

Theorem 0.1.52 ***Lagrange remainder*** *For* $n \in \mathbb{N}$ *and* $x > 0$ *there exists* $c \in (0, x)$ *such that*

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

if $x < 0$*, then* $c \in (x, 0)$

Note: *c depends on both n and x*

Proof: fix $x > 0$ and consider

$$h(t) = f(t) - s_n(t) - \left(\frac{f(x) - s_n(x)}{x^{n+1}} \right) t^{n+1}$$

Note that:

$$h(x) = 0 \quad \text{ and } \quad h^{(k)}(0) = 0, \, k = 0, ..., n$$

The lemma gives $c \in (0, x)$ such that

$$f^{(n+1)}(c) - s_n^{(n+1)}(c) - (n+1)! \left(\frac{f(x) - s_n(x)}{x^{n+1}} \right) = 0$$

Rearraging gives

$$f(x) - s_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

Taylor series around different points

Assume f is inf. often diff.ble on interval around a

Definition 0.1.28 *The Taylor series of f around $x = a$ is given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Theorem 0.1.53 *For $x > a$ there exists $c \in (a, x)$ such that*

$$E_n(x) = f(x) - s_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

if $x < a$ then $c \in (x, a)$

The Riemann Integral

The Fundamental Theorem of Calculus is a statement about the inverse relationship between differentiation and integration. It comes in two parts, depending on whether we are differentiating an integral or integrating a derivative. The Fundamental Theorem of Calculus states that:

- $\int_a^b F'(x)dx = F(b) - F(a)$ and
- if $G(x) = \int_a^x f(t)dt$ then $G'(x) = f(x)$

Nevertheless, for understand it completely we need first to define Partition, Upper Sums, and Lower Sums:

Definition 0.1.29 ***Partitions*** *A partitions of $[a, b]$ is a set of the form*

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

Let $f : [a, b] \to \mathbb{R}$ be bounded and P be a partition of $[a, b]$

Definition 0.1.30 ***Lower sum*** *Lower sum of f w.r.t P*

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ L(f, P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be bounded and P be a partition of $[a, b]$

Definition 0.1.31 ***Upper sum*** *Upper sum of f w.r.t P*

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} \\ U(f, P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \end{aligned}$$

Note: For a particular partition P, it is clear that $U(f, P) \geq L(f, P)$

Definition 0.1.32 ***Refinements*** *Q is called a refinement of P if $P \subset Q$. Provided that P and Q are partitions of the same interval.*

Lemma 0.1.5 *If $P \subset Q$ then*

$$L(f, P) \leq L(f, Q) \quad \text{and} \quad U(f, P) \geq U(f, Q)$$

Corollary 0.1.33 *If $P \subset Q$ then*

$$U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P)$$

Proof (lower sum) Lemma 4.3.4: refine P by adding one point $z \in [x_{k-1}, x_k]$

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ m'_k &= \inf\{f(x) : x \in [z, x_k]\} \\ m"_k &= \inf\{f(x) : x \in [x_{k-1}, z]\} \end{aligned}$$

Remember that $A \subset B$ then $\inf A \geq \inf B$

$$\begin{aligned} m_k(x_k - x_{k-1}) &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m'_k(x_k - z) + m"_k(z - x_{k-1}) \end{aligned}$$

Then proceed by induction

Lemma 0.1.6 *for two partitions P_1 and P_2 we have $L(f, P_1) \leq U(f, P_2)$*

Proof: let $Q = P_1 \cup P_2$ then $P_1, P_2 \subset Q$ so

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2)$$

Integrability

Assume $f : [a, b] \to \mathbb{R}$ is bounded

Let $\mathcal{P}$ denote the collection of all partitions of $[a, b]$

Definition 0.1.34 *The upper integral of f is defined to be*

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

The lower integral of f by

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

Lemma 0.1.7 *For any bounded function f on $[a, b]$, it is always the case that $U(f) \geq L(f)$*

Definition 0.1.35 *A bounded function $f : [a, b] \to \mathbb{R}$ is called **Rimann integrable** if $U(f) = L(f)$*

Notation:

$$\int_a^b f = U(f) = L(f) \quad \text{or} \quad \int_a^b f(x)dx = U(f) = L(f)$$

Theorem 0.1.54 ***Criterion of integrability*** *The following statements are equivalent*

(i). f is integrable

(ii). for all $\epsilon > 0$ there exists a partition P_ϵ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

Proof (2$\Rightarrow$ 1) :

$$\begin{cases} U(f) \leq U(f, P_\epsilon) \\ L(f) \geq L(f, P_\epsilon) \end{cases} \Rightarrow U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$$

This holds for all $\epsilon > 0$ so $U(f) = L(f)$

Proof (1$\Rightarrow$ 2): *let $\epsilon > 0$ and choose P_1 and P_2 such that*

$$L(f, P_1) > L(f) - \frac{1}{2}\epsilon \quad \textit{and} \quad U(f, P_2) < U(f) + \frac{1}{2}\epsilon$$

Let $P_\epsilon = P_1 \cup P_2$ then

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq U(f, P_2) - L(f, P_1) \\ &= [U(f, P_2) - U(f)] + [L(f) - L(f, P_1)] \\ &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon \\ &= \epsilon \end{aligned}$$

Theorem 0.1.55 *f continuous on $[a, b] \Rightarrow f$ is integrable on $[a, b]$*

Proof: *f is uniformly continuous on $[a, b]$*

For all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\epsilon}{b - a} \quad \textit{for all } x, y \in [a, b]$$

Let P be a partition such that $x_k - x_{k-1} < \delta$ for all $k = 1, 2, ..., n$

There exist $y_k, z_k \in [x_{k-1}, x_k]$ such that

$$f(y_k) = M_k \quad \textit{and} \quad f(z_k) = m_k$$

Note:

$$|y_k - z_k| < \delta \quad \Rightarrow \quad M_k - m_k = f(y_k) - f(z_k) < \frac{\epsilon}{b-a}$$

Thus

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{\epsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= \frac{\epsilon}{b-a}\cdot(x_n - x_0) \\ &= \frac{\epsilon}{b-a}\cdot(b-a) = \epsilon \end{aligned}$$

Example: *any increasing function* $f : [a,b] \to \mathbb{R}$ *is integrable*

For any partition of $[a,b]$ *we have*

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k\} \\ &= f(x_k) \end{aligned}$$

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k\} \\ &= f(x_{k-1}) \end{aligned}$$

An equispaced partition P *gives*

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) \\ &= \frac{(b-a)}{n}\sum_{k=1}^{n}[f(x_k) - f(x_{k-1}] \\ &= \frac{(b-a)(f(b) - f(a)}{n} \to 0 \qquad as\ n \to \infty \end{aligned}$$

Properties of integrals

Theorem 0.1.56 Split property *Let* $f : [a,b] \to \mathbb{R}$ *be bounded and* $c \in (a,b)$, *then*

$$f \ \ integrable\ on\ [a,b] \quad \Leftrightarrow \quad f \ \ integrable\ on\ [a,c] \ \ and\ [c,b]$$

In that case

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof ($\Rightarrow$): *Let $\epsilon > 0$ and pick a partition P of $[a,b]$ s.t.*

$$U(f,P) - L(f,P) < \epsilon$$

Let $P_c = P \cup \{c\}$ then

$$U(f,P_c) - L(f,P_c) < \epsilon$$

Then $Q = P_c \cap [a,c]$ is a partition of $[a,c]$ and

$$\left.\begin{array}{lll} m & := & \#\ \textit{intervals in } Q \\ n & := & \#\ \textit{intervals in } P_c \end{array}\right\} \Rightarrow m < n$$

$m < n$ implies

$$\begin{aligned} U(f,Q) - L(f,Q) &= \sum_{k=1}^{m} (M_k - m_k)(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= U(f,P_c) - L(f,P_c) \\ &< \epsilon \end{aligned}$$

Conclusion: f is integrable on $[a,c]$. The proof for $[c,b]$ is similar. Proof ($\Leftarrow$): *Let P_1 and P_2 be partitions of $[a,c]$ and $[c,b]$ s.t*

$$U(f,P_i) - L(f,P_i) < \frac{1}{2}\epsilon, \quad i = 1,2$$

Then $P = P_1 \cup P_2$ is a partition of $[a,b]$ and

$$\begin{aligned} U(f,P) &= U(f,P_1) + U(f,P_2) \\ L(f,P) &= L(f,P_1) + L(f,P_2) \\ U(f,P) - L(f,P) &< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \end{aligned}$$

Conclusion: f is integrable on $[a,b]$

Let ϵ and P_1 and P_2 be as before

$$\begin{aligned} \int_a^b f &\leq U(f,P) \\ &< L(f,P) + \epsilon \\ &= L(f,P_1) + L(f,P_2) + \epsilon \\ &\leq \int_a^c f + \int_c^b f + \epsilon \end{aligned}$$

$$\int_a^b f \leq \int_a^c f + \int_c^b f$$

Let ϵ and P_1 and P_2 be as before

$$\begin{aligned}\int_a^c f + \int_c^b f &\le U(f,P_1) + U(f,P_2)\\ &< L(f,P_1) + L(f,P_2) + \epsilon\\ &= L(f,P) + \epsilon\\ &\le \int_a^b f + \epsilon\end{aligned}$$

$$\int_a^c f + \int_c^b f \le \int_a^b f$$

And we have done.

Definition 0.1.36 *if f is integrable on $[a.b]$ then*

$$\int_a^b f = -\int_b^a f \quad \text{and} \quad \int_c^c f = 0 \text{ for all } c \in \mathbb{R}$$

Theorem 0.1.57 *if f, g are integrable on $[a,b]$ then*

- *$f+g$ integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$*
- *kf integrable and $\int_a^b kf = k\int_a^b f$ for all $k \in \mathbb{R}$*

Theorem 0.1.58 *If f is integrable on $[a,b]$ then*

$$m \le f(x) \le M \Rightarrow m(b-a) \le \int_a^b f \le M(b-a)$$

Proof: *for all partitions P of $[a,b]$*

$$L(f,P) \le \int_a^b f \le U(f,P)$$

Taking $P = \{a,b\}$ gives

$$\begin{aligned}U(f,P) &= (b-a)\cdot \sup\{f(x) : x \in [a,b]\} \le M(b-a)\\ L(f,P) &= (b-a)\cdot \inf\{f(x) : x \in [a,b]\} \ge m(b-a)\end{aligned}$$

Theorem 0.1.59 *if f, g are integrable on $[a,b]$ then*

$$f(x) \le g(x) \quad \text{for all } x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$$

Proof: *since $0 \le g(x) - f(x)$ for all $x \in [a,b]$ we have*

$$0\cdot(b-a) \le \int_a^b (g-f) \Rightarrow 0 \le \int_a^b g - \int_a^b f$$

Theorem 0.1.60 *If f is integrable on $[a,b]$ then $|f|$ is integrable and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Proof: *Let P be any partition of $[a,b]$ and*

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} \\ m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ M_k' &= \sup\{|f(x)| : x \in [x_{k-1}, x_k]\} \\ m_k' &= \inf\{|f(x)| : x \in [x_{k-1}, x_k]\} \end{aligned}$$

claim: $M_k' - m_k' \leq M_k - m_k$

For all $\epsilon > 0$ there exist $y, z \in [x_{k-1}, x_k]$ s.t

$$\begin{aligned} M_k' - \frac{1}{2}\epsilon &< |f(y)| \\ m_k' + \frac{1}{2}\epsilon &> |f(z)| \end{aligned}$$

$$\begin{aligned} M_k' - m_k' - \epsilon &< |f(y)| - |f(z)| \\ &\leq |f(y) - f(z)| \\ &\leq M_k - m_k \end{aligned}$$

$$M_k' - m_k' \leq M_k - m_k$$

Let P any partition of $[a,b]$ then

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{k=1}^{n} (M_k' - m_k')(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \\ &= U(f, P) - L(f, P) \end{aligned}$$

Thus,

$$\begin{aligned} -|f(x)| \leq f(x) \leq |f(x)| &\Rightarrow -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \\ &\Rightarrow \left|\int_a^b f\right| \leq \int_a^b |f| \end{aligned}$$

The fundamental theorem of calculus

Theorem 0.1.61 FTC part 1 *assume that*

(i). f is integrable on $[a, b]$

(ii). F is differentiable on $[a, b]$ and

$$F'(x) = f(x) \quad \forall x \in [a, b]$$

Then

$$\int_a^b f = F(b) - F(a)$$

Proof: *let P be any partition of $[a, b]$*

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})) \\ \textit{By the MVT } &= \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \qquad t_k \in (x_{k-1}, x_k) \\ &\leq \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \\ &= U(f, P) \\ &\geq L(f, P) \end{aligned}$$

let P be any partition of $[a, b]$, then

$$L(f, P) \leq F(b) - F(a) \leq U(f, P)$$

Taking sup/inf over all partitions gives

$$L(f) \leq F(b) - F(a) \leq U(f)$$

Since f is integrable it follows that

$$L(f) = U(f) = F(b) - F(a)$$

Theorem 0.1.62 FTC part 2 *let f be integrable on $[a, b]$ and define*

$$F(x) = \int_a^x f(t)dt \quad \textit{where } x \in [a, b]$$

Then

(i). F is uniformly continuous on $[a, b]$

(ii). if f is continuous at c, then F is differentiable at c and

$$F'(c) = f(c)$$

Proof(1) *since f is integrable on $[a, b]$ there exists $M > 0$ s.t.*

$$|f(x)| \leq M \quad \forall x \in [a, b]$$

If $x, y \in [a, b]$ with $x \geq y$, then

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_y^x f(t)dt \right| \\ &\leq \int_y^x |f(t)|dt \\ &\leq M|x - y| \end{aligned}$$

For given $\epsilon > 0$ take $\delta = \epsilon / M$.

Proof(2): *for $x \neq c$ we have*

$$\begin{aligned} \frac{F(x) - F(c)}{x - c} - f(c) &= \frac{1}{x - c} \int_c^x f(t)dt - f(c) \\ &= \frac{1}{x - c} \int_c^x f(t) - f(c)dt \end{aligned}$$

Let $\epsilon > 0$ be arbitrary and pick $\delta > 0$ s.t

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$$

Since $|t - c| \leq |x - c| < \delta$ it follows that

$$\begin{aligned} \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \frac{1}{|x - c|} \left| \int_c^x f(t) - f(c)dt \right| \\ &\leq \frac{1}{|x - c|} \cdot |x - c| \cdot \epsilon \\ &= \epsilon \end{aligned}$$